A Nonlinear Dynamical Theory for Heterogeneous, Anisotropic, Elastic Rods

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A large deformation, small-strain theory is presented for heterogeneous, transverse isotropic, elastic rods with pre-twist. The theory is applicable to practical problems related to the dynamics of cable systems, helicopter blades, space antennae, and similar structures. Two elementary examples are included: reduction of the general theory to particular differential equations governing the planar, steady-state towing of cables, and the steady-state motion of helicopter rotor blades.

I. Introduction

IN this paper a large deformation, small-strain theory is formulated for heterogeneous, anisotropic, elastic rods. The resulting model allows approximate treatment of practical problems related to the dynamics of cable systems, helicopter blades, space antennae, and similar structures. Equations governing the planar steady-state towing of cables, and the steady-state motion of helicopter rotor blades are obtained from the general theory as elementary example applications to specific problems. A more comprehensive application of the theory to faired-cable stability is discussed in a sequel to this work.

The development commences from three-dimensional considerations and proceeds as follows: In Sec. II, a reference system is introduced and certain necessary kinematic relations are derived. In Sec. III, a rod displacement field is postulated, and associated rod strains are obtained in terms of the kinematic variables of Sec. II; the kinematic variables are subject to constraints that allow flexure according to the Kirchhoff hypothesis, cross-sectional warping or axial torsion in the sense of St. Venant, and axial extension. Next, classical resultant forces and moments are defined in Sec. IV by appropriate averaging of stresses over the rod cross section. By use of a linear elastic constitutive relation, a premise concerning the relative magnitude of certain stress components, and the results of Secs. II and III, the resultant force and moment fields are constrained to satisfy D'Alembert's principle. The latter yields six scalar equations of motion (Sec. V). These, together with relations from the previous sections, result in 16 equations for 16 unknowns: 3 components each of velocity, curvature, spin, force, and moment vectors, and the strain of the rod reference curve. Sec. VI is a summary of these equations. Depending upon the problem, the actual number of equations necessary may vary considerably. Examples concerning cables and helicopter blades are provided in Secs. VII and VIII, respectively, to illustrate this point.

The rod theory presented herein is more general than the classical Kirchhoff[1] theory (c.f., Ref. 2, Chaps. 18 and 19). Similarities include the equations of equilibrium in the absence of dynamics (given by Clebsch, contained in the work of Kirchhoff). Differences include the following: 1) the concept of the "shear center" and its effect on the rod constitutive relations is absent in Refs. 1 and 2, but is contained in the present theory (relative locations of the shear center, the aerodynamics center, and the centroid of the axial stress field constitute critical parameters in stability analyses of both helicopter blades and faired cables); 2) kinematic restraints in Refs. 1 and 2 render the theory inappropriate for helicopter-blade problems – these constraints are relaxed in the present work; 3) heterogeneity and anisotropy are contained in the present theory but are absent in Refs. 1 and 2 (most faired-cables are both heterogeneous and anisotropic); 4) in contrast to Refs. 1 and 2, the reference curve in the current theory is arbitrary. In addition, the methods of derivation differ considerably – the present derivation is transparent.

On the other hand, it is noted that more sophisticated rod models currently exist. However, examination of each such model reveals features that render it impractical for the aforementioned class of problems. Several examples should suffice to illustrate this point.

In Ref. 3, Hay considered the finite displacement of thin rods. Hay based his theory upon expansions explicitly in terms of the smallness parameter, ratio of the "diameter" to characteristic "rod length." As Antman and Warner[4] have correctly noted, the approach in Ref. 3 does not consider contact loads on the lateral surface of the rod. Consequently, the model is limited in terms of practical applications.

Beginning with the Kirchhoff assumptions, but including extension of the reference axis, cross-sectional warping, and shear deformation, Antman[5] has formulated an "exact" fully nonlinear theory for *one-dimensional* elasticity, where the constitutive relations are given in general, admissible forms. No attempt, however, is made to relate these constitutive relations to three-dimensional constitutive equations of either linear or nonlinear elasticity. In view of practical applications, this necessitates a series of experiments to determine the one-dimensional constitutive relations as suggested by Reissner.[6]

Antman and Warner[4] have conducted an extensive study of hyperelastic rods; in particular, a hierarchy of approximating theories have been obtained from three-dimensional elasticity. Theories of various orders are obtained by appropriate "projections" on function spaces. The infinite-order theory, although intractable, is formally exact. Each *n*th-order theory is determinate. However, interpretation of the dependent variables, and hence the resulting equations, may pose a problem in practical applications. For example, in classical rod theories, stress measures consist of the three components of the stress-resultant vector and the three components of the couple-resultant vector. In Ref. 4, a set of stress measures is formed from weighted averages of the three-dimensional stress tensor.

For further general treatment of the rod problem, and associated discussion, the reader is referred to the articles of

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Green and Laws,[7] Green, Laws, and Naghdi,[8] and the excellent extensive review article by Antman.[9]

II. Geometrical Preliminaries

In portions of the subsequent analysis, indicial notations[10,11] will be employed. Latin indices range over 1, 2, 3; Greek indices range over 1, 2. The summation convention holds in each case.

Consider a rod with slowly varying cross-sectional dimensions, and let $\mathcal{C}_0$ (Fig. 1) denote the locus of material particles at similar cross-sectional locations. A particle of $\mathcal{C}_0$ has the position vector

$$r_0 = r_0(s_0) \tag{1}$$

where s_0 denotes distance along $\mathcal{C}_0$. In the deformed state, the locus of these same particles is a curve $\mathcal{C}$. At time t, a particle that initially was a distance s_0 on $\mathcal{C}_0$ is a distance s on $\mathcal{C}$, and has the position vector

$$r = r(s,t) \tag{2}$$

If the strain e of $\mathcal{C}$ is defined by

$$2e = (ds^2 - ds_0^2)/ds_0^2 = |\partial r/\partial s|^2 - 1 \tag{3}$$

then the variables s_0 and s are related by

$$ds = \sqrt{1+2e}\, ds_0 \tag{4}$$

To each point of $\mathcal{C}$, three mutually orthogonal unit vectors $A_i(s,t)$ are now assigned as follows: The vector A_3 is the unit tangent vector to $\mathcal{C}$, defined by

$$A_3(s,t) = \partial r(s,t)/\partial s \tag{5}$$

and the vectors A_α are orthogonal to A_3. For a composite rod, we consider the constituent materials to be transversely isotropic and the axis of symmetry to be aligned locally with the A_3. Cartesian coordinates θ_α now are introduced in the plane normal to A_3, and a weighted inertia tensor $\overline{EI}_{\alpha\beta}$ is defined by

$$\overline{EI}_{\alpha\beta} = \int_A\int E(\theta_1,\theta_2)\theta_\alpha\theta_\beta d\theta_1 d\theta_2 \tag{6}$$

where E represents the Young's modulus in the A_3 direction. The unit vectors A_α may be chosen to be parallel or coincident with the principal axes of $\overline{EI}_{\alpha\beta}$.

In addition, let a set of vectors a_i be assigned to the initial curve $\mathcal{C}_0$ through

$$a_i(s_0) = A_i(s_0,0) \tag{7}$$

From the definitions of A_i, a_i, it is evident that

$$A_i \cdot A_j = a_i \cdot a_j = \delta_{ij} \tag{8}$$

where δ_{ij} is the Kronecker delta.

Consider next differentiation of $A_i(s,t)$ with respect to s. If, following Eq. (8), the curvatures κ_i of $\mathcal{C}$ are defined according to

$$\kappa_\alpha = A_\alpha \cdot A_3' = -A_3 \cdot A_\alpha' \qquad \kappa_3 = A_2 \cdot A_1' = -A_1 \cdot A_2' \tag{9}$$

where $(\ )' \equiv \partial(\ )/\partial s$, then

$$A_1' = \kappa_3 A_2 - \kappa_1 A_3 \qquad A_2' = \kappa_3 A_1 - \kappa_2 A_3 \qquad A_3' = \kappa_1 A_1 - \kappa_2 A_2 \tag{10}$$

In addition to Eq. (10), it will be necessary to compute the *material* time derivatives

$$\dot{A}_i \equiv dA_i[s(s_0,t),t]/dt \tag{11}$$

If the components Ω_i of the spin vector Ω are defined by

$$\Omega_1 = \dot{A}_2 \cdot A_3 = -\dot{A}_3 \cdot A_2 \qquad \Omega_2 = \dot{A}_3 \cdot A_1 = -\dot{A}_1 \cdot A_3 \qquad \Omega_3 = \dot{A}_1 \cdot A_2 = -\dot{A}_2 \cdot A_1 \tag{12}$$

then

$$\dot{A}_1 = \Omega_3 A_2 - \Omega_2 A_3 \qquad \dot{A}_2 = -\Omega_3 A_1 + \Omega_1 A_3 \qquad \dot{A}_3 = \Omega_2 A_1 - \Omega_1 A_2 \tag{13}$$

Now, let $\dot{r}$ denote the velocity vector of a particle on $\mathcal{C}$. It will be convenient to represent $\dot{r}$ in the form

$$\dot{r} = V_i A_i \tag{14}$$

From Eqs. (10) and (14), one finds

$$(\dot{r})' = (V_1' - \kappa_3 V_2 + \kappa_1 V_3)A_1 + (V_2' + \kappa_3 V_1 + \kappa_2 V_3)A_2 + (V_3' - \kappa_1 V_1 - \kappa_2 V_2)A_3 \tag{15}$$

On the other hand, from Eqs. (5) and (4),

$$\dot{A}_3 = (r')^{\cdot} = (\dot{r})' - \dot{e}(1+2e)^{-1}A_3 \tag{16}$$

Upon combining Eqs. (15) and (16), and comparing the result with the last of Eq. (13), the following expressions for Ω_i are obtained:

$$-\Omega_1 = V_2' + \kappa_3 V_1 + \kappa_2 V_3 \qquad \Omega_2 = V_1' - \kappa_3 V_2 + \kappa_1 V_3 \qquad \dot{e}(1+2e)^{-1} = V_3' - \kappa_1 V_1 - \kappa_2 V_2 \tag{17}$$

Finally, consider the material time derivative of the curvatures. From Eq. (9),

$$\dot{\kappa}_3 = (A_2 \cdot A_1')^{\cdot} = \dot{A}_2 \cdot A_1' + A_2 \cdot [\dot{A}_1' - \dot{e}(1+2e)^{-1}A_1'] \tag{18}$$

Substitution of Eqs. (10, 13, and 17) into Eq. (18) yields

$$\dot{\kappa}_3 = \Omega_3' - \Omega_1\kappa_1 - \Omega_2\kappa_2 - \dot{e}(1+2e)^{-1}\kappa_3 \tag{19a}$$

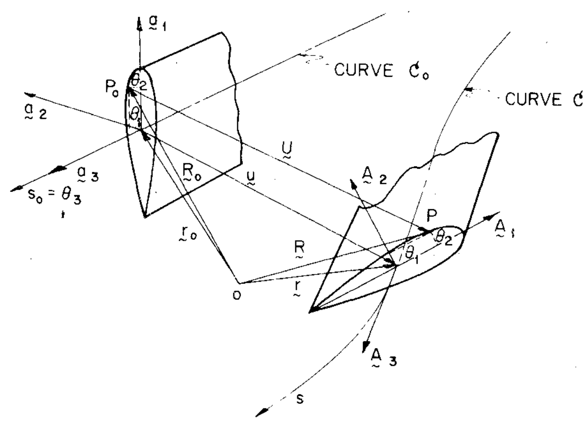

Fig. 1 Coordinate system and displacement field.

In a similar manner, the first of Eq. (9) furnishes

$$\dot{\kappa}_1 = \Omega_2' + \Omega_3\kappa_2 + \Omega_1\kappa_3 - \dot{e}(1+2e)^{-1}\kappa_1$$
$$\dot{\kappa}_2 = -\Omega_1' - \Omega_3\kappa_1 + \Omega_2\kappa_3 - \dot{e}(1+2e)^{-1}\kappa_2 \qquad (19b)$$

In view of the foregoing, it is evident that all kinematic quantities associated with $\mathcal{C}$ can be considered as functions of four dependent variables: $V_i\,(i=1,2,3)$ and Ω_3.

III. Displacement and Strain Fields

Metric Tensor of Undeformed State

The position vector $\boldsymbol{R}_0$ of a particle in the undeformed rod is represented in the form

$$\boldsymbol{R}_0(\theta_1,\theta_2,\theta_3) = \boldsymbol{r}_0(\theta_3) + \theta_\alpha \boldsymbol{a}_\alpha(\theta_3) \qquad (20)$$

where $\theta_3 \equiv s_0$ and θ_α denote distances along the $\boldsymbol{a}_\alpha$ axes. In the following it will be assumed that in the underformed configuration the reference curve $\mathcal{C}_0$ is a straight line and the rod is twisted about this line. Let η_3 represent the twist per unit length of $\mathcal{C}_0$, then

$$\partial \boldsymbol{a}_1/\partial s_0 = \eta_3 \boldsymbol{a}_2 \qquad \partial \boldsymbol{a}_2/\partial s_0 = -\eta_3 \boldsymbol{a}_1 \qquad (21)$$

Consequently, the base vectors $\boldsymbol{g}_i = \partial \boldsymbol{R}_0/\partial\theta_i$ associated with the coordinates θ_i are given by

$$\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha \qquad \boldsymbol{g}_3 = \boldsymbol{a}_3 + \eta_3(\theta_1\boldsymbol{a}_2 - \theta_2\boldsymbol{a}_1) \qquad (22)$$

The components of the metric tensor $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ associated with the undeformed state then follow as

$$g_{\alpha\beta} = \delta_{\alpha\beta} \qquad g_{13} = -\eta_3\theta_2 \qquad g_{23} = \eta_3\theta_1$$
$$g_{33} = 1 + \eta_3^2(\theta_1^2 + \theta_2^2) \qquad [g \equiv \det(g_{ij}) = 1] \qquad (23)$$

Displacement Field

A fundamental premise regarding the kinematics of the rod is now made. The position vector $\boldsymbol{R}$ of a particle in the deformed state (originally at $\boldsymbol{R}_0$ in the undeformed state) is written‡

$$\boldsymbol{R}(\theta_1,\theta_2,\theta_3,t) = \boldsymbol{r}(\theta_3,t) + \theta_\alpha \boldsymbol{A}_\alpha(\theta_3,t) + \kappa(\theta_3,t)\varphi(\theta_1,\theta_2)\boldsymbol{A}_3(\theta_3,t) \qquad (24)$$

Here φ is St. Venant's warping function[12] and κ is the effective twist per unit length of the undeformed reference curve $\mathcal{C}_0$, namely

$$\kappa = \sqrt{1+2e}\,\kappa_3 - \eta_3 \qquad (25)$$

Under Eq. (24), the displacement vector $\boldsymbol{U}$ (Fig. 1) is given by

$$\boldsymbol{U} = \boldsymbol{R} - \boldsymbol{R}_0 = \boldsymbol{u} + \theta_\alpha(\boldsymbol{A}_\alpha - \boldsymbol{a}_\alpha) + \kappa\varphi\boldsymbol{A}_3 \qquad (26a)$$

where

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{r}_0 \qquad (26b)$$

is the displacement of a particle on the reference curve $\mathcal{C}_0$, $\theta_\alpha(\boldsymbol{A}_\alpha - \boldsymbol{a}_\alpha)$ represents a displacement due to cross-sectional rotation, and $\kappa\varphi\boldsymbol{A}_3$ represents cross-sectional warping due to the twist κ.

‡θ_i are undeformed (Lagrangian) coordinates in Eq. (24); they are not convected.

Warping Function φ

A few comments concerning the warping function φ are in order at this point. For a homogeneous cross section A in the undeformed state, φ satisfies Laplace's equation

$$\partial^2\varphi/\partial\theta_1^2 + \partial^2\varphi/\partial\theta_2^2 = 0 \qquad (27)$$

in A, and

$$\partial\varphi/\partial n = \theta_2(\boldsymbol{n}\cdot\boldsymbol{A}_1) - \theta_1(\boldsymbol{n}\cdot\boldsymbol{A}_2) \qquad (28)$$

on the boundary Γ of the region A; here $\boldsymbol{n}$ is the outer normal to A.

Consider now a rod with a heterogeneous cross section A. Let this cross section be divided into N regions $A^{(i)}$, $i=1, 2, \ldots, N$. Within each region let the material properties be linearly elastic, transversely isotropic, and homogeneous (some regions may be void). Further, let the bonds between adjacent nonvoid regions be perfect. The material properties are thus piecewise constant functions of given cross-sectional coordinates.

Let $A^{(i)}$, $i=1, 2, \ldots, k$ denote nonvoid subregions of A, and $A^{(i)}$, $i=k+1, \ldots, N$ the void regions. In addition, let $\boldsymbol{n}$ denote the outer normal to the lateral rod surface and $\boldsymbol{n}^{(i)}$ represent outward normals to the boundaries $\Gamma^{(i)}$ of $A^{(i)}$ (the regions $A^{(i)}$ may be multiply connected, thus the boundaries $\Gamma^{(i)}$ may be disconnected). Then, an extension of St. Venant's free torsion solution to composite cross sections of the type under discussion reveals that φ is defined by the following: 1) $\varphi = \varphi^{(i)}$ in $A^{(i)}$, where $\varphi^{(i)}$ satisfies Eq. (27) in $A^{(i)}$, $i=1, 2, \ldots, k$; 2) φ is single-valued and continuous throughout the nonvoid portion of A; 3) φ satisfies Eq. (28) on Γ; 4) $\varphi^{(i)}$ satisfy the jump conditions

$$\lim_{\substack{n^{(i)} \to \Gamma^{(i)} \\ n^{(i)} \in A^{(i)}}} [G^{(i)}\partial\varphi^{(i)}/\partial n^{(i)}] + \lim_{\substack{n^{(j)} \to \Gamma^{(j)} \\ n^{(j)} \in A^{(j)}}} [G^{(j)}\partial\varphi^{(j)}/\partial n^{(j)}] = (G^{(i)} - G^{(j)})[\theta_2(\boldsymbol{n}^{(i)}\cdot\boldsymbol{A}_1) - \theta_1(\boldsymbol{n}^{(i)}\cdot\boldsymbol{A}_2)]_{\text{on}\,\Gamma^{(i)}} \qquad (29)$$

across that portion of $\Gamma^{(i)}$ adjacent to a nonvoid region $A^{(j)}$, $j=1, 2, \ldots, k$. If a portion of $\Gamma^{(i)}$ is adjacent to a void region $A^{(j)}$, $j=k+1, \ldots, N$, then $\varphi^{(i)}$ must satisfy Eq. (29), with $G^{(j)}=0$. In the preceding, $G^{(i)}$ denotes the shear modulus of the material in region $A^{(i)}$, and $n^{(i)}$ is a distance coordinate in the direction of $\boldsymbol{n}^{(i)}$. No summation over indices is implied.

Metric Tensor of Deformed State

With respect to the undeformed coordinates θ_i, the base vectors $\boldsymbol{G}_i = \partial\boldsymbol{R}/\partial\theta_i$ of the deformed state are, with the use of Eqs. (24, 10, and 4),

$$\boldsymbol{G}_\alpha = \boldsymbol{A}_\alpha + \kappa\varphi_{,\alpha}\boldsymbol{A}_3$$
$$\boldsymbol{G}_3 = \sqrt{1+2e}\{[\kappa_1\kappa\varphi - \kappa_3\theta_2]\boldsymbol{A}_1 + [\kappa_2\kappa\varphi + \kappa_3\theta_1]\boldsymbol{A}_2 + [1 - \theta_\alpha\kappa_\alpha + \varphi\partial\kappa/\partial s]\boldsymbol{A}_3\} \qquad (30)$$

where $\varphi_{,\alpha} \equiv \partial\varphi/\partial\theta_\alpha$.

The metric tensor of the deformed state $G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j$ is computed from Eq. (30) as

$$G_{\alpha\beta} = \delta_{\alpha\beta} + \kappa^2\varphi_{,\alpha}\varphi_{,\beta}$$
$$G_{13} = \sqrt{1+2e}[\kappa\varphi_{,1}(1 - \theta_\alpha\kappa_\alpha + \kappa'\varphi) + \kappa_1\kappa\varphi - \kappa_3\theta_2]$$
$$G_{23} = \sqrt{1+2e}[\kappa\varphi_{,2}(1 - \theta_\alpha\kappa_\alpha + \kappa'\varphi) + \kappa_2\kappa\varphi + \kappa_3\theta_1] \qquad (31)$$
$$G_{33} = (1+2e)[(1 - \theta_\alpha\kappa_\alpha + \kappa'\varphi)^2 + (\kappa_1\kappa\varphi - \kappa_3\theta_2)^2 + (\kappa_2\kappa\varphi + \kappa_3\theta_1)^2]$$

where $\kappa' \equiv \partial\kappa/\partial s$.

Strains under "Thin-Rod" Approximation

If Green's strain tensor[10] γ_{ij} is employed as a measure of deformation, we have

$$2\gamma_{ij}=G_{ij}-g_{ij} \tag{32}$$

where the components of γ_{ij} are referred to the coordinates θ_i. It will facilitate portions of the analysis if the components of the strain tensor are referred to local rectangular Cartesian coordinates Y_1, Y_2, Y_3 along A_1, A_2, A_3, respectively (here $Y_\alpha=\theta_\alpha$). Upon denoting the new components by e_{ij}, we obtain

$$e_{ij}=\gamma_{rs}\frac{\partial\theta_r}{\partial Y_i}\frac{\partial\theta_s}{\partial Y_j} \tag{33}$$

From the geometrical relation

$$d\boldsymbol{R}=\boldsymbol{G}_i d\theta_i=\boldsymbol{A}_i dY_i \tag{34}$$

we find

$$\partial\theta_i/\partial Y_j=\boldsymbol{G}^i\cdot\boldsymbol{A}_j \tag{35}$$

where the contravariant base vectors $\boldsymbol{G}^i$ are defined in terms of the covariant base vectors $\boldsymbol{G}_i$ by

$$\sqrt{G}e_{rst}\boldsymbol{G}^t=\boldsymbol{G}_r\times\boldsymbol{G}_s \tag{36}$$

The quantity e_{rst} in Eq. (36) is the permutation symbol, and G denotes $\det(G_{ij})$.

Under the approximations,

$$e,\qquad \gamma_{ij},\qquad b\kappa_i,\qquad b^2\partial\kappa/\partial s\ll 1 \tag{37}$$

($b=\max|\theta_\alpha|$), the nonzero components of e_{ij} are obtained, from use of Eqs. (23 and 31-36), as

$$\begin{aligned}2e_{13}&=\kappa[\varphi_{,1}+\kappa_1\varphi-\theta_2]\\ 2e_{23}&=\kappa[\varphi_{,2}+\kappa_2\varphi+\theta_1]\\ e_{33}&=e+\kappa'\varphi-\theta_\alpha\kappa_\alpha\end{aligned} \tag{38a}$$

where

$$\kappa=\sqrt{1+2e}\kappa_3-\eta_3\doteq\kappa_3-\eta_3 \tag{39}$$

In most cases of interest, it also can be assumed that $\varphi_{,\alpha}\gg\varphi\kappa_\alpha$, and thus the first two of Eq. (38a) simplify to

$$2e_{13}=\kappa(\varphi_{,1}-\theta_2)\qquad 2e_{23}=\kappa(\varphi_{,2}+\theta_1) \tag{38b}$$

Equations (38a) and (38b) constitute a small strain, large deflection "thin-rod" theory. As is evident from Eq. (37), the theory requires that the ratio of the maximum cross-sectional dimension to radius of curvature be small as compared to unity.

In the case of an inhomogeneous rod, it is evident from the discussion of the warping function that $e_{\alpha3}$ are piecewise continuous functions in the cross section A.

IV. Stress Resultants and Constitutive Relations

Stress Tensor

Let the components of the Cauchy stress tensor σ^{ij} be referred to the coordinates Y_1, Y_2, Y_3. Since these coordinates are rectangular Cartesian, the position of the indices are immaterial, and we shall simply write σ_{ij}. Under the assumption that the rod materials are linearly elastic, homogeneous within each subregion $A^{(i)}$ of A, and transversely isotropic with the plane of symmetry normal to A_3, the constitutive relations are taken in the form

$$e_{\alpha\beta}=\frac{1+\upsilon^*}{E^*}\sigma_{\alpha\beta}-\frac{\upsilon^*}{E^*}\sigma_{\gamma\gamma}\delta_{\alpha\beta}-\frac{\upsilon}{E}\sigma_{33}\delta_{\alpha\beta}$$

$$e_{\alpha3}=\frac{\sigma_{\alpha3}}{2G},\quad e_{33}=\frac{\sigma_{33}-\upsilon\sigma_{\alpha\alpha}}{E} \tag{40}$$

where E^*, υ^* represent the Young's modulus and the Poisson's ratio in the plane of isotropy (i.e., the plane of A_1, A_2) and E, υ represent the corresponding constants in the A_3 direction, G is the shear modulus. We note that in the case of transverse isotropy G is an independent elastic constant.

An approximation now is introduced; it will be assumed that the stresses $\sigma_{\alpha\beta}$ can be neglected in comparison to σ_{33} in Eq. (40). As a consequence, the following relations are obtained for e_{i3}:

$$e_{33}\doteq\sigma_{33}/E\qquad e_{\alpha3}=\sigma_{\alpha3}/2G \tag{41}$$

Stress Resultants

Let us define resultant forces N_i and resultant moments M_i by

$$N_i=\int_A\int\sigma_{3i}dA\qquad i=1,2,3 \tag{42a}$$

$$M_1=\int_A\int\theta_2\sigma_{33}dA,\qquad M_2=-\int_A\int\theta_1\sigma_{33}dA$$

$$M_3=\int_A\int[\theta_1\sigma_{32}-\theta_2\sigma_{31}]dA \tag{42b}$$

where $dA\equiv d\theta_1 d\theta_2$.

Substitution of Eq. (41) into (42a), with the use of Eq. (38), furnishes the following relations between N_3, M_α, and e, κ_α:

$$\begin{aligned}N_3&=\overline{EA}(e-\theta_\alpha^{(n)}\kappa_\alpha)+\overline{EA}_\varphi\kappa'\\ M_1&=\overline{EA}\theta_2^{(n)}e+P_2\kappa'-\overline{EI}_{22}\kappa_2\\ M_2&=-\overline{EA}\theta_1^{(n)}e-P_1\kappa'+\overline{EI}_{11}\kappa_1\end{aligned} \tag{43a}$$

where, with $E=E(\theta_1,\theta_2)$, $\varphi=\varphi(\theta_1,\theta_2)$,

$$\overline{EA}=\int_A\int EdA,\ \overline{EA}_\varphi=\int_A\int E\varphi dA,\ \overline{EI}_{\alpha\beta}=\int_A\int E\theta_\alpha\theta_\beta\alpha A$$

$$P_\alpha=\int_A\int\theta_\alpha E\varphi dA,\ \theta_\alpha^{(n)}=\int_A\int E\theta_\alpha dA\Big/\int_A\int EdA \tag{43b}$$

The quantities $\theta_1^{(n)}$, $\theta_2^{(n)}$ define the locations of the neutral axes for bending under moments M_2, M_1, respectively. The equation for M_3 deserves careful attention. In view of the assumed displacement field, the strains $e_{\alpha3}$ are only due to St. Venant torsion, i.e., Eq. (38) do not reflect shearing strains due to transverse loads (this is typical of a Bernoulli-Euler bending approximation). Let the exact strains $e_{\alpha3}^e$ be decomposed as follows:

$$e_{\alpha3}^e=e_{\alpha3}+e_{\alpha3}^* \tag{44}$$

where $e_{\alpha3}$ are given by Eq. (38b) and $e_{\alpha3}^*$ denote the contributions due to transverse loads. Substituting Eq. (44) into (42b), we find that

$$M_3=\overline{GJ}\kappa+N_2\theta_1^{(s)}\text{-}N_1\theta_2^{(s)} \tag{45a}$$

where, with $G = G(\theta_1, \theta_2)$,

$$\overline{GJ} = \int_A \int G(\theta_1 \varphi_{,2} - \theta_2 \varphi_{,1} + \theta_1^2 + \theta_2^2)\, dA$$

$$\theta_1^{(s)} = \int_A \int \sigma_{32}^* \theta_1 dA \Big/ \int_A \int \sigma_{32}^* dA$$

$$\theta_2^{(s)} = \int_A \int \sigma_{31}^* \theta_2 dA \Big/ \int_A \int \sigma_{31}^* dA, \quad \sigma_{3\alpha}^* = 2Ge_{3\alpha}^* \tag{45b}$$

The quantities $\theta_\alpha^{(s)}$ in Eq. (45b) define the "shear center" of the rod cross section. It will be assumed that $\theta_\alpha^{(s)}$ are slowly varying functions of θ_3, and that they can be computed from the standard cantilever case. This is equivalent to the assumption that the distribution of $\sigma_{3\alpha}^*$ with respect to θ_α is similar for all θ_3.

In the derivation of Eq. (45b), the fact that $\sigma_{3\alpha}$ do not give rise to transverse resultant shear forces was employed, i.e.,

$$\int_A \int G e_{3\alpha} dA = 0 \tag{46}$$

Consider, for example, e_{31}:

$$\int_A \int G(\theta_1, \theta_2)(\varphi_{,1} - \theta_2) d\theta_1 d\theta_2 = \sum_i \int_A \int G_i (\varphi_{,1}^{(i)} - \theta_2) dA$$

$$= \sum_i \int_{A_i} \int G_i \{ [\theta_i (\varphi_{,1}^{(i)} - \theta_2)]_{,1} + [\theta_1 \varphi_{,2}^{(i)} + \theta_1]_{,2} \} d\theta_1 d\theta_2$$

Applying Gauss' theorem to each integral, and noting that $\varphi^{(i)}$ must be single-valued and satisfy Eq. (27), one obtains

$$= \sum_i G_i \int_{\Gamma_i} \theta_i [\partial \varphi^{(i)} / \partial n^{(i)} - \theta_2 (\boldsymbol{n}^{(i)} \cdot \boldsymbol{A}_1) + \theta_1 (\boldsymbol{n}^{(i)} \cdot \boldsymbol{A}_2)] d\Gamma_i$$

which vanishes because of Eq. (29).

Symmetries

If the rod cross section has an axis of symmetry, of say, the $\boldsymbol{A}_2$ axis, then $E(\theta_1, \theta_2) = E(-\theta_1, \theta_2)$, $\varphi(\theta_1, \theta_2) = -\varphi(-\theta_1, \theta_2)$, which yields

$$\theta_1^{(n)} = \theta_1^{(s)} = P_2 = \overline{EA}\varphi = 0 \tag{47}$$

If both axes are axes of symmetry, and $\mathcal{C}_0$ is selected as the locus of centroids of A, then

$$\theta_\alpha^{(n)} = \theta_\alpha^{(s)} = P_\alpha = \overline{EA}\varphi = 0 \tag{48}$$

In general, some simplification results from a judicious choice of $\mathcal{C}_0$. For example, if $\mathcal{C}_0$ is selected along the neutral axes for bending about the $\boldsymbol{A}_\alpha$ axes, then $\theta_\alpha^{(n)} = 0$. On the other hand, if $\mathcal{C}_0$ is selected as the locus of material points along the cross-sectional shear center, then $\theta_\alpha^{(s)} = 0$.

An Additional Approximation

If a typical wavelength of rod motion is l, then the terms in Eq. (43a) involving $\partial \kappa / ds$ are $O(b^2/l^2)$, whereas the remaining terms are $O(b/l)$, where $b = \max |\theta_\alpha|$. Thus, for sufficiently long wavelengths l, Eq. (43a) can be simplified to read

$$N_3 = \overline{EA}(e - \theta_\alpha^{(n)} \kappa_\alpha)$$

$$M_1 = \overline{EA}\theta_2^{(n)} e - \overline{EI}_{22}\kappa_2, \quad M_2 = -\overline{EA}\theta_1^{(n)} e + \overline{EI}_{11}\kappa_1 \tag{49}$$

V. Equations of Motion

Let us define the external resultant force and moment vectors $\boldsymbol{f}^{(E)}$, $\boldsymbol{m}^{(E)}$ in terms of the traction vector $\boldsymbol{T}^{(n)}$ on the lateral surface of the rod according to

$$\boldsymbol{f}^{(E)} = \int_\Gamma \boldsymbol{T}^{(n)} d\Gamma, \qquad \boldsymbol{m}^{(E)} = \int_\Gamma \boldsymbol{\theta} \times \boldsymbol{T}^{(n)} d\Gamma \tag{50}$$

where $\boldsymbol{\theta} = \theta_\alpha \boldsymbol{A}_\alpha$. In addition, let us define body and inertial force and moment resultants by

$$\boldsymbol{f}^{(I)} = -\int_A \int \rho \ddot{\boldsymbol{R}} dA, \quad \boldsymbol{m}^{(I)} = -\int_A \int \rho \boldsymbol{\theta} \times \ddot{\boldsymbol{R}} dA$$

$$\boldsymbol{f}^{(B)} = \int_A \int \rho \boldsymbol{B} dA, \qquad \boldsymbol{m}^{(B)} = \int_A \int \rho \boldsymbol{\theta} \times \boldsymbol{B} dA \tag{51}$$

where ρ denotes density and $\boldsymbol{B}$ the body force per unit mass of the rod.

Substitution of Eq. (24) into (51) furnishes

$$-\boldsymbol{f}^{(I)} = m\ddot{\boldsymbol{r}} + m\theta_\alpha^{(g)} \ddot{\boldsymbol{A}}_\alpha + m_\varphi \ddot{\overline{\rho \boldsymbol{A}_3}} \tag{52a}$$

$$-\boldsymbol{m}^{(I)} = m\theta_\alpha^{(g)} \boldsymbol{A}_\alpha \times \ddot{\boldsymbol{r}} + I_{\alpha\beta} \boldsymbol{A}_\alpha \times \ddot{\boldsymbol{A}}_\beta + \Theta_\alpha \boldsymbol{A}_\alpha \times (\ddot{\overline{\kappa \boldsymbol{A}_3}}) \tag{52b}$$

where, with $\rho = \rho(\theta_1, \theta_2)$,

$$m = \int_A \int \rho dA, \quad m_\varphi = \int_A \int \rho\varphi dA, \quad \Theta_\alpha = \int_A \int \theta_\alpha \rho \varphi dA$$

$$I_{\alpha\beta} = \int_A \int \theta_\alpha \theta_\beta \rho dA, \quad \theta_\alpha^{(g)} = \int_A \int \theta_\alpha \rho dA \Big/ \int_A \int \rho dA \tag{53}$$

Symmetries

If geometrical and material symmetries exist about, say, the θ_2 axis, then

$$I_{12} = I_{21} = \theta_1^{(g)} = m_\varphi = \Theta_1 = 0 \tag{54}$$

If symmetry exists about both the θ_1 and θ_2 axes, then, in addition to Eq. (54), we have

$$\theta_2^{(g)} = \Theta_\alpha = 0 \tag{55}$$

Approximations

It can be shown that in Eq. (52) the $\ddot{\overline{\kappa A}}$ term is $O(b/l)$ compared to the remaining terms. Thus, if $b/l \ll 1$, the last terms in Eq. (52) can be neglected. This approximation will be adopted in all subsequent work. We note that this approximation, $b/l \ll 1$, is in fact consistent with our assumed displacement field, and when $b/l = O(1)$, transverse shear deformation, which is not included in Eq. (24), becomes important.

According to our averaging procedure, an element ds of the rod is subjected to resultant forces and moments, as indicated in Fig. 2. Applying D'Alembert's principle, these resultant

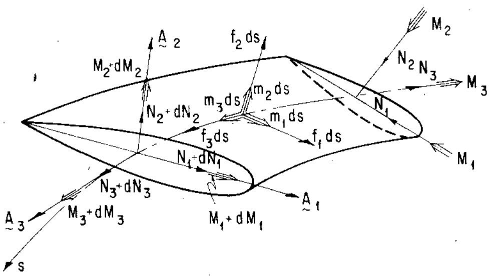

Fig. 2 Resultant force and moment field on element of length ds.

force and moment fields are required to vanish. This furnishes the following equations of motion:

$$\partial N/\partial s+f=0, \qquad f\equiv f^{(E)}+f^{(I)}+f^{(B)}$$

$$\partial M/\partial s+A_3\times N+m=0, \quad m\equiv m^{(E)}+m^{(I)}+m^{(B)} \qquad (56)$$

VI. Summary of Basic Equations

Let all vector quantities F be decomposed as follows§:

$$F=F_iA_i \qquad (57)$$

Then, Eqs. (56, 43, 16, and 18) constitute 16 equations for 16 unknowns: N_i, M_i, V_i, κ_i,Ω_i, and e. These basic rod differential equations are summarized in the following.

Conservation of Linear and Angular Momentum

$$N_1'-\kappa_3N_2+\kappa_1N_3+f_1=0 \qquad (58a)$$

$$M_1'-\kappa_3M_2+\kappa_1M_3-N_2+m_1=0 \qquad (58b)$$

$$N_2'+\kappa_3N_1+\kappa_2N_3+f_2=0 \qquad (59a)$$

$$M_2'+\kappa_3M_1+\kappa_2M_3+N_1+m_2=0 \qquad (59b)$$

$$N_3'-\kappa_1N_1-\kappa_2N_2+f_3=0 \qquad (60a)$$

$$M_3'-\kappa_1M_1-\kappa_2M_2+m_3=0 \qquad (60b)$$

Constitutive Relations

$$N_3=\overline{EA}(e-\theta_\alpha^{(n)}\kappa_\alpha) \qquad (61a)$$

$$M_3=\overline{GJ}(\kappa_3-\eta_3)+N_2\theta_1^{(s)}-N_1\theta_2^{(s)} \qquad (61b)$$

$$M_1=\overline{EA}\theta_2^{(n)}e-\overline{EI}_{22}\kappa_2 \qquad (62a)$$

$$M_2=-\overline{EA}\theta_1^{(n)}e+\overline{EI}_{11}\kappa_1 \qquad (62b)$$

Geometric Relations

$$\dot\kappa_1=-\dot e\kappa_1+\Omega_3\kappa_2+\Omega_1\kappa_3+\Omega_2'$$

$$\dot\kappa_2=-\dot e\kappa_2-\Omega_3\kappa_1+\Omega_2\kappa_3-\Omega_1'$$

$$\dot\kappa_3=-\dot e\kappa_3-\Omega_1\kappa_1-\Omega_2\kappa_2+\Omega_3' \qquad (63)$$

$$\dot e=V'_3-\kappa_1V_1-\kappa_2V_2$$

$$\Omega_1=-V'_2-\kappa_3V_1-\kappa_2V_3$$

$$\Omega_2=V'_1-\kappa_3V_2+\kappa_1V_3 \qquad (64)$$

where $(\ )'\equiv\partial(\ )/\partial\theta_3$ or $\partial(\ )/\partial s$, since $e\ll 1$.

Intertial Forces

Employing Eqs. (12) and (52), the inertial forces and moments can be written in component form as

$$f_1^{(I)}=m[\Omega_3V_2-\Omega_2V_3-\dot V_1]+m\theta_2^{(g)}(\dot\Omega_3-\Omega_1\Omega_2)$$
$$+m\theta_1^{(g)}(\Omega_3^2+\Omega_2^2)$$

$$f_2^{(I)}=m[\Omega_1V_3-\Omega_3V_1-\dot V_2]-m\theta_1^{(g)}(\dot\Omega_3+\Omega_1\Omega_2)$$
$$+m\theta_2^{(g)}(\Omega_1^2+\Omega_3^2)$$

$$f_3^{(I)}=m[\Omega_2V_1-\Omega_1V_2-\dot V_3]+m\theta_1^{(g)}(\dot\Omega_2-\Omega_3\Omega_1)$$
$$-m\theta_2^{(g)}(\dot\Omega_1+\Omega_2\Omega_3) \qquad (65a)$$

§This decomposition is often the most convenient for incremental computational schemes.

$$m_1^{(I)}=m\theta_2^{(g)}(\Omega_2V_1-\Omega_1V_2-\dot V_3]-I_{22}(\dot\Omega_1+\Omega_2\Omega_3)$$
$$+I_{12}(\dot\Omega_2-\Omega_3\Omega_1)$$

$$m_2^{(I)}=m\theta_1^{(g)}(\Omega_1V_2-\Omega_2V_1+\dot V_3]-I_{11}(\dot\Omega_2-\Omega_1\Omega_3)$$
$$+I_{12}(\dot\Omega_1+\Omega_3\Omega_2)$$

$$m_3^{(I)}=m\theta_1^{(g)}(\Omega_1V_3-\Omega_3V_1-\dot V_2)$$
$$+m\theta_2^{(g)}(\Omega_2V_3-\Omega_3V_2+\dot V_1)$$
$$-I_{11}(\dot\Omega_3+\Omega_1\Omega_2)-I_{22}(\dot\Omega_3-\Omega_1\Omega_2)\ -I_{12}(\Omega_2^2-\Omega_1^2)$$
$$(65b)$$

In the following sections, we deduce the appropriate governing equations in the case of two applications namely, 1) planar steady-state towing of cables, and 2) steady-state motion of helicopter blades.

VII. Planar Steady-State Towing of Cables

The governing equations in the case of planar steady-state towing of cables with cross-sectional symmetry about the tow-plane (Fig. 3) can be obtained from our rod equations (58-65) as follows.

Selecting the locus of material points along the neutral axis, $\theta_1^{(n)}=\theta_2^{(n)}=0$, as the reference curve $\mathcal{C}$, we define planar steady-state by 1) all variables in Eqs. (58-65) are independent of time, and 2) the following conditions hold:

$$\kappa_2=\kappa_3=0, \qquad N_2=M_1=M_3=0$$
$$f_2=m_i=0, \qquad \Omega_i=V_i=0 \qquad (66)$$

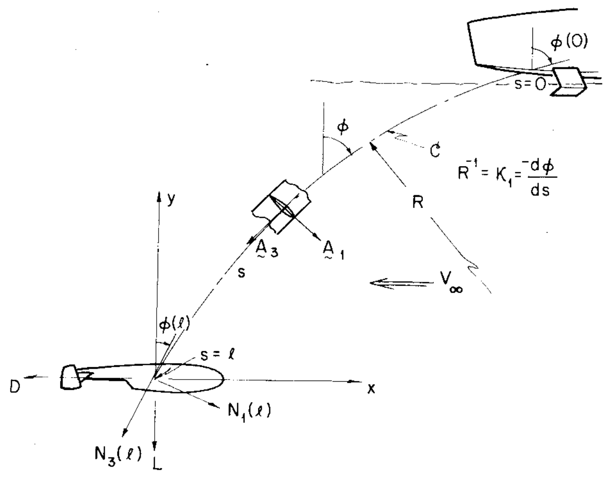

Fig. 3 Steady-state planar towing configuration.

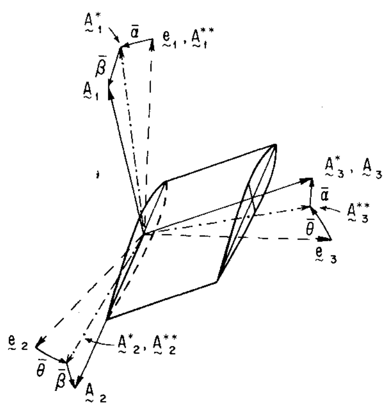

Fig. 4 Coordinate system and geometry of a helicopter blade.

Under items 1 and 2 in the preceding, Eqs. (58) reduce to

$$dN_1/ds+\kappa_1 N_3+f_1^{(E)}+f_1^{(B)}=0 \qquad (67a)$$

$$dN_3/ds-\kappa_1 N_1+f_3^{(E)}+f_3^{(B)}=0 \qquad (67b)$$

$$dM_2/ds+N_1=0 \qquad (67c)$$

The constitutive relations (61) become

$$N_3=\overline{EA}(e-\theta_1^{(n)}\kappa_1) \qquad (68a)$$

$$M_2=\overline{EI}_{11}\kappa_1-\overline{EA}\theta_1^{(n)}e \qquad (68b)$$

Equations (67) and (68) are to be supplemented by appropriate boundary conditions of the form

$$M_2(0)=M_2(l)=0$$

$$N_1(l)=L\sin\varphi(l)-D\cos\varphi(l)$$

$$N_3(l)=L\cos\varphi(l)+D\sin(l) \qquad (69)$$

where D, L denote drag and downward lift forces exerted on the lower cable terminus by the towed body, Fig. 3; φ is the angle between the vertical and local tangent to $\mathcal{C}$, and l is the undeformed cable length. Equations (69) imply a ball or hinge joint at the upper and lower cable termini.

The solutions of Eqs. (67) and (68), and their local stability criteria in the form of divergence and flutter analyses, will be presented in a sequel to this paper.

VIII. Steady-State Equation of Motion for Helicopter Rotor Blades

As a second application of the nonlinear rod equations (58-65) we consider the steady-state motion of a nonuniform, twisted helicopter rotor blade. The geometry and coordinate systems necessary to describe the rotor blade are shown in Fig. 4.

Three mutually orthogonal unit vectors e_i describe the inertial frame of reference. With respect to this frame, a unit triad A_i^{**} rotates with an angular velocity $\Omega A_i^{**}\equiv\Omega e_1$. Selecting the neutral axis of the rotor blade as the reference curve $\mathcal{C}$, we place the triad A_i^{**} at r, such that

$$A_i^{**}=\theta_{ij}e_j \quad \text{or} \quad e_i=\theta_{ji}A_j^{**} \qquad (70a)$$

where

$$\theta_{1i}=\theta_{i1}=\delta_{1i},\ \theta_{22}=\theta_{33}=\cos\bar\theta \qquad (70b)$$

$$\theta_{23}=-\theta_{32}=\sin\bar\theta,\ \bar\theta\equiv\Omega t-\theta(s) \qquad (70c)$$

with $\theta(s)$ measuring the bending deformation of the rotor blade about the A_1^{**} axis.

Next, we introduce a third triad A_i^*, with the following relation to A_i^{**}:

$$A_i^*=\alpha_{ij}A_j^{**} \quad \text{or} \quad A_i^{**}=\alpha_{ji}A_j^* \qquad (71a)$$

where

$$\alpha_{2i}=\alpha_{i2}=\delta_{i2},\quad \alpha_{11}=\alpha_{33}=\cos\bar\alpha \qquad (71b)$$

$$\alpha_{31}=-\alpha_{13}=\sin\bar\alpha,\quad \bar\alpha=\alpha_0+\alpha(s) \qquad (71c)$$

Here α_0 is the pre-cone angle of the rotor blades with e_2, e_3 reference plane, and $\bar\alpha$ is the local value of this angle after deformation.

Finally, the orthogonal unit vectors A_i are introduced in such a way that $A_3^*=A_3$ and A_1 and A_2 are, respectively, normal and coincident with the axis of symmetry of the deformed cross section. We define the angle of twist $\bar\beta$ to relate A_i and A_i^* as follows

$$A_i=\beta_{ij}A_j^* \quad \text{or} \quad A_i^*=\beta_{ji}A_j \qquad (72a)$$

where

$$\beta_{3i}=\beta_{i3}=\delta_{i3},\quad \beta_{11}=\beta_{22}=\cos\bar\beta \qquad (72b)$$

$$\beta_{12}=-\beta_{21}=\sin\bar\beta,\quad \bar\beta=\beta_0(s)+\beta(s) \qquad (72c)$$

where $\beta_0(s)$ represents the pre-twist angle of the rotor blade.

Combining Eqs. (70-72), we write

$$e_i=\theta_{ji}\alpha_{kj}\beta_{\ell k}A_\ell \qquad (73)$$

Setting $\dot e_i=0$, we obtain the material derivative of A_i as

$$\dot A_i=-\beta_{ij}\alpha_{jk}\theta_{kl}\dot\theta_{ml}\alpha_{nm}\beta_{pn}A_p\dot\theta_{ij}=\Omega d\theta_{ij}/d\bar\theta \qquad (74)$$

Use of Eqs. (70-72) in Eq. (74), and comparison with Eq. (13) furnishes

$$\Omega_3=\Omega\sin\bar\alpha,\ \ \Omega_2=-\Omega\sin\bar\beta\cos\bar\alpha,\ \ \Omega_1=\Omega\cos\bar\beta\cos\bar\alpha \qquad (75)$$

Setting the spacial derivatives of e_i in Eq. (73) to zero, we obtain A_i', which upon comparison with Eq. (10), gives

$$\kappa_3=\frac{d\beta}{ds}-\sin\bar\alpha\frac{d\theta}{ds}$$

$$\kappa_2=\cos\bar\alpha\cos\bar\beta\frac{d\theta}{ds}-\sin\bar\beta\frac{d\alpha}{ds}$$

$$\kappa_1=\cos\bar\beta\frac{d\alpha}{ds}+\cos\bar\alpha\sin\bar\beta\frac{d\theta}{ds} \qquad (76)$$

If we consider $\alpha(s)$, $\beta(s)$, and $\theta(s)$ as the appropriate measures of deformation, then Eq. (76) represents three nonlinear, nonhomogeneous differential equations for these angles. From the constitutive relations, (61) and (62), we have

$$\kappa_1=M_2/\overline{EI}_{11},\qquad \kappa_2=-M_1/\overline{EI}_{22}$$

$$\kappa_3=\frac{d\beta_0}{ds}+\frac{M_2+N_1\theta_2^{(s)}-N_2\theta_1^{(s)}}{\overline{GJ}},\qquad e=\frac{N_3}{\overline{EA}} \qquad (77)$$

In order to complete this formulation, Eqs. (75-77) are to be supplemented with the six momentum equations (58-60), where the inertial forces now are defined by

$$f_1^{(I)}=m[\Omega_3V_2-\Omega_2V_3]-m\theta_2^{(g)}\Omega_1\Omega_2$$

$$f_2^{(I)}=m[\Omega_1V_3-\Omega_3V_1]+m\theta_2^{(g)}(\Omega_1^2+\Omega_3^2)$$

$$f_3^{(I)}=m[\Omega_2V_1-\Omega_1V_2]-m\theta_2^{(g)}\Omega_2\Omega_3 \qquad (78a)$$

$$m_1^{(I)}=m\theta_2^{(g)}[\Omega_2V_1-\Omega_1V_2]-I_{22}\Omega_2\Omega_3$$

$$m_2^{(I)}=I_{11}\Omega_1\Omega_3$$

$$m_3^{(I)}=m\theta_2^{(g)}[\Omega_2V_3-\Omega_3V_2]+(I_{22}-I_{11})\Omega_1\Omega_2 \qquad (78b)$$

and where, from the symmetry of the cross section, we have set $\theta_1^{(g)}=0$.

In Eq. (78) the components of velocity V_i are related to Ω_i through the differential equations (17), namely,

$$V_2'+\kappa_3V_1+\kappa_2V_3=-\Omega_1,\qquad V_1'-\kappa_3V_2+\kappa_1V_3=\Omega_2$$

$$V_3'-\kappa_1V_1+\kappa_2V_2=0 \qquad (79)$$

We note that, in integrating the system of equations (79), the velocity of the inertial frame of reference, that is, the steady-state velocity of the helicopter, is to be incorporated through appropriate boundary conditions. Equations (75-77, 58-60, and 79) constitute 19 equations for the 19 unknown quantities α, β, θ, Ω_i, κ_i, N_i, M_i, V_i, and e. The nonlinear nature of these equations would require an incremental computational scheme wherein the only independent variable s is to be discretized appropriately for numerical integrations. The boundary conditions associated with these equations and the aerodynamic loading functions will not be discussed here. A detailed study of these equations and the stability of their solutions (numerical as well as dynamic) will be considered in a later work.

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